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Calculation of Singular Extremal Rocket Trajectories

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The equations determining singular intermediate-thrust extremal arcs in an inverse square law gravitational field, for which a rocket's propellant expenditure is stationary with respect to small variations, are transformed into a form more convenient for numerical computation in the general three-dimensional case. It is shown that the set of essentially distinguishable arcs is doubly infinite and the types of behavior of the rocket thrust along these trajectories are described. A comparison is made between various orbital transfers effected via these arcs and optimal two-impulse transfers.

Introduction

If r is the position vector of a rocket in some inertial frame and v its velocity, then its equations of motion in a gravitational field of intensity $g(r,t)$ at time t are

$$\dot{r} = v, \qquad \dot{v} = g + fu \tag{1}$$

where f is the acceleration imparted by the motor and u the unit vector in the direction of thrust. The characteristic velocity C for a maneuver extending over a time span (t_0, t_1) is given by

$$C = \int_{t_0}^{t_1} f \, \mathrm{d}t \tag{2}$$

To render C stationary with respect to weak variations of the propellant expenditure program, it is known[1] that it is necessary to solve adjoint equations for the primer vector p and its derivative q in the form

$$\dot{p} = q, \qquad \dot{q} = \nabla(p \cdot g) \tag{3}$$

For any realizable field, there exists a potential function ϕ such that

$$g = -\nabla\phi \tag{4}$$

Whenever the rocket motor is operative (i.e., $f \neq 0$), it is a requirement for stationarity that the primer be of constant magnitude and that it be aligned with the thrust. If f is unconstrained, this may lead to the application of impulsive thrusts at (what are termed) junction points. However, it has been proved[2] that solutions to Eqs. (1) and (3) can always be found for which the magnitude of p is constant and f is finite. Such intermediate-thrust (or singular) arcs can, accordingly, form part of a trajectory for which C is stationary.

If the times of commencement and termination of the maneuver (t_0, t_1) are not prescribed, but are available for variation (time-open case), then at these instants it is necessary that

$$p \cdot g - q \cdot v = 0 \tag{5}$$

If, in addition, g is not explicitly dependent on t (steady field), then Eq. (5) is valid over the whole trajectory, being a first integral of Eqs. (1) and (3).

In the case of an inverse square law field with $g = -\gamma r/r^3$, provided that motion is confined to a plane through the center of attraction, a complete set of intermediate-thrust arcs, described by elementary functions, can be found.[3] In the time-open case, these arcs take the form of spirals about the center of attraction as a pole, the polar coordinates (r,θ) of points on the curves being given by the equations

$$r = a \sin^6\psi/(1 - 3\sin^2\psi), \qquad \theta = \alpha - 4\psi - 3\cot\psi \tag{6}$$

where a, α are parameters labeling members of the doubly infinite family and ψ is the angle between the thrust and the perpendicular to the radius r.

It is known[4-9] that, for the time-open case, intermediate-thrust arcs cannot form part of a trajectory minimizing C. Nevertheless, it is desirable that their form should be established and a comparison made between the characteristic velocity for this type of maneuver and that required by an optimal impulsive transfer. The purpose of this paper is to investigate the calculation of IT arcs (in the time-open case) when the motion is not confined to a plane and to effect some comparisons with optimal two-impulse transfers.

Inverse Square Law Equations

If p, q satisfy Eqs. (3), then so do αp, αq, where α is a constant multiplier. It follows that if p has constant magnitude, we can assume that this magnitude is unity and derive the general solution later by restoring the numerical factor. Thus, we shall take

$$p^2 = 1 \tag{7}$$

and $u = p$.

If $g = -\gamma r/r^3$, then

$$\nabla(p \cdot g) = \frac{3\gamma}{r^5}(p \cdot r)r - \frac{\gamma}{r^3} p \tag{8}$$

Thus, in an inverse square law field, we have to solve the following equations:

$$\dot{r} = v, \qquad \dot{v} = -\frac{\gamma r}{r^3} + fp \tag{9a}$$

$$\dot{p} = q, \qquad \dot{q} = \frac{3\gamma}{r^5}(p \cdot r)r - \frac{\gamma}{r^3} p \tag{9b}$$

subject to the constraint (7).

By differentiating Eq. (7) with respect to time, we can derive a number of auxiliary equations. Thus,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(p^2) = p \cdot q = 0 \tag{10}$$

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(p^2) = \dot{p} \cdot q + p \cdot \dot{q} = 0 \tag{11}$$

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or

$$q^2 + \frac{3\gamma}{r^5}(p\cdot r)^2 - \frac{\gamma}{r^3} = 0 \tag{12}$$

Differentiation of Eq. (12) gives

$$2q\cdot\dot{q} - \frac{15\gamma}{r^6}\dot{r}(p\cdot r)^2 + \frac{6\gamma}{r^5}(p\cdot r)(\dot{p}\cdot r + p\cdot\dot{r}) + \frac{3\gamma}{r^4}\dot{r} = 0 \tag{13}$$

Since $r\cdot\dot{r} = \mathrm{d}(\frac{1}{2}r^2)/\mathrm{d}t = \mathrm{d}(\frac{1}{2}r^2)/\mathrm{d}t = r\dot{r}$, and $\dot{r}$, $\dot{p}$, $\dot{q}$ can be substituted for from Eqs. (9), Eq. (13) is found to be equivalent to

$$(r\cdot v)\left[1 - \frac{5}{r^2}(p\cdot r)^2\right] + 2(p\cdot r)(2q\cdot r + p\cdot v) = 0 \tag{14}$$

A further differentiation of Eq. (14) and substitution for $\dot{r}$, $\dot{p}$, $\dot{q}$, $\dot{v}$ yields the following equation for the thrust acceleration f:

$$\begin{aligned}(p\cdot r)\left[3 - \frac{5}{r^2}(p\cdot r)^2\right]f &= -2(q\cdot r + p\cdot v)(2q\cdot r + p\cdot v)\\ &- 6(p\cdot r)(q\cdot v) - \left[1 - \frac{5}{r^2}(p\cdot r)^2\right]v^2 + \frac{\gamma}{r} - \frac{11\gamma}{r^3}(p\cdot r)^2\\ &+ \frac{10}{r^2}(p\cdot r)(v\cdot r)\left[q\cdot r + p\cdot v - \frac{1}{r^2}(v\cdot r)(p\cdot r)\right]\end{aligned} \tag{15}$$

In the time-open case we are studying, the first integral (5) is applicable and this has the form

$$q\cdot v + \frac{\gamma}{r^3}p\cdot r = 0 \tag{16}$$

A further equation that proves to be useful is obtainable thus

$$\frac{\mathrm{d}}{\mathrm{d}t}(q\times r - p\times v) = \dot{q}\times r - p\times\dot{v} = 0 \tag{17}$$

by Eqs. (9). A first integral

$$q\times r - p\times v = A \tag{18}$$

where A is a constant vector now follows immediately.

To perform a numerical integration of our equations, we need to choose initial values satisfying Eqs. (7), (10), (12), (14), and (16). The following procedure could be used to generate such values: A unit vector p and a vector q perpendicular to it are first chosen arbitrarily, thus satisfying Eqs. (7) and (10). Next, we choose a unit vector n and suppose that $r = rn$. Then Eq. (12) requires that

$$\frac{\gamma}{r^3} = q^2/\left[1 - 3(p\cdot n)^2\right] \tag{19}$$

thus fixing r; note that n is arbitrary, except that it must satisfy the condition $(p\cdot n)^2 < \frac{1}{3}$. The position vector r has now been determined. Equation (14) takes the form

$$a\cdot v = b \tag{20}$$

with a and b being known quantities. Finally, v can be found to satisfy this equation and Eq. (16) in ∞^1 ways.

Altogether seven arbitrary scalar quantities have to be prescribed during the procedure described in the preceding paragraph. Accordingly, ∞^7 IT arcs will be generated. However, since any point on such an arc may be regarded as the initial point and the field has spherical symmetry about the center of attraction, only ∞^3 of these arcs are essentially distinct. Further, it may be verified that if

$$r = F(t),\quad v = G(t),\quad p = H(t),\quad q = J(t),\quad f = K(t) \tag{21}$$

satisfy all of our equations, then so do

$$r = \alpha F(\beta t),\quad v = \alpha^{-\frac{1}{2}}G(\beta t),\quad p = H(\beta t)$$
$$q = \alpha^{-3/2}J(\beta t),\quad f = \alpha^{-2}K(\beta t) \tag{22}$$

where $\beta = \alpha^{-3/2}$. The trajectory corresponding to the second of these solutions is identical to the trajectory corresponding to the first after this has been magnified by the factor α. Hence, only one of these trajectories needs to be computed as a representative of the set and this reduces the family of distinguishable trajectories to a doubly infinite set.

Having computed a possible set of starting values for r, v, p, q, we have shown in the foregoing reference[2] that the integration of Eqs. (9), using the value of f determined by Eq. (15), will result in values for these vector variables that continue to satisfy Eqs. (7), (10), (12), (14), (16), and (18) for later values of t. It is possible, therefore, to ignore the equations for $\dot{v}$ and $\dot{q}$ in Eqs. (9) and to integrate $\dot{r} = v$, $\dot{p} = q$ for r and p, Eqs. (10), (12), (14), (16), and (18) being used to fix v and q. However, owing to the complexity of these equations, such a derivation of v and q would be numerically difficult. Instead, we shall first refer the equations to a frame that rotates with the primer when, as explained in the next section, the integration can be carried through with greater facility.

Reference Frame Rotating with the Primer

Equation (10) shows that q is invariably perpendicular to p. It follows that the vectors p, q, $p\times q$ constitute a right-handed Cartesian reference frame R, along the axes of which all of our vectors can be resolved into components. Thus, we shall take

$$r = Xp + Yq + Zp\times q \tag{23}$$

$$v = Up + Vq + Wp\times q \tag{24}$$

Since q and $p\times q$ are not unit vectors, but both have magnitude q, the components of r and v along the axes of R are (X, qY, qZ) and (U, qV, qW), respectively. Hence,

$$r^2 = X^2 + q^2(Y^2 + Z^2) \tag{25}$$

$$v^2 = U^2 + q^2(V^2 + W^2) \tag{26}$$

It will be helpful, at this stage, to note the following scalar products:

$$p\cdot r = X,\quad q\cdot r = q^2Y,\quad (p\times q)\cdot r = q^2Z \tag{27}$$

$$p\cdot v = U,\quad q\cdot v = q^2V,\quad (p\times q)\cdot v = q^2W \tag{28}$$

and the following vector triple products:

$$p\times(p\times q) = -q,\quad q\times(p\times q) = q^2p \tag{29}$$

Note also that

$$p\times r = -Zq + Yp\times q,\quad q\times r = q^2Zp - Xp\times q \tag{30}$$

$$p\times v = -Wq + Vp\times q,\quad q\times v = q^2Wp - Up\times q \tag{31}$$

The time derivatives of the base vectors for R will be derived next. We have $\dot{p} = q$ and, from Eqs. (9),

$$\dot{q} = \frac{\gamma}{r^3}\left(\frac{3X^2}{r^2} - 1\right)p + \frac{3\gamma}{r^5}X(Yq + Zp\times q) \tag{32}$$

Thus,

$$\frac{d}{dt}(p \times q) = \dot{p} \times q + p \times \dot{q} = \frac{3\gamma}{r^5}(p \cdot r)(p \times r)$$

$$= \frac{3\gamma}{r^5} X(-Zq + Yp \times q) \tag{33}$$

We can now differentiate Eq. (23) to give

$$\dot{r} = \dot{X}p + \dot{Y}q + \dot{Z}p \times q + X\dot{p} + Y\dot{q} + Z\frac{d}{dt}(p \times q)$$

$$= \left(\dot{X} + \frac{3\gamma}{r^5}X^2Y - \frac{\gamma}{r^3}Y\right)p + \left(\dot{Y} + X + \frac{3\gamma}{r^5}XY^2 - \frac{3\gamma}{r^5}XZ^2\right)q + \left(\dot{Z} + \frac{6\gamma}{r^5}XYZ\right)p \times q \tag{34}$$

Since $\dot{r} = v$, this implies that

$$\dot{X} + \frac{3\gamma}{r^5}X^2Y - \frac{\gamma}{r^3}Y = U \tag{35}$$

$$\dot{Y} + X + \frac{3\gamma}{r^5}X(Y^2 - Z^2) = V \tag{36}$$

$$\dot{Z} + \frac{6\gamma}{r^5}XYZ = W \tag{37}$$

Similarly, differentiating Eq. (24) and comparing the result with Eq. (9b) in the form

$$\dot{v} = \left(f - \frac{\gamma}{r^3}X\right)p - \frac{\gamma}{r^3}Yq - \frac{\gamma}{r^3}Zp \times q \tag{38}$$

we conclude that

$$\dot{U} + \frac{\gamma}{r^3}\left(X - V + \frac{3X^2}{r^2}V\right) = f \tag{39}$$

$$\dot{V} + U + \frac{\gamma}{r^3}\left[Y + \frac{3X}{r^2}(YV - ZW)\right] = 0 \tag{40}$$

$$\dot{W} + \frac{\gamma}{r^3}\left[Z + \frac{3X}{r^2}(ZV + YW)\right] = 0 \tag{41}$$

The first integrals (12), (14), (16), and (18) are next seen to be equivalent to the equations

$$q^2 = \frac{\gamma}{r^3}\left(1 - \frac{3X^2}{r^2}\right) \tag{42}$$

$$X\left(\frac{5X^2}{r^2} - 3\right)U = q^2\left[\left(1 - \frac{5X^2}{r^2}\right)(YV + ZW) + 4XY\right] \tag{43}$$

$$q^2V = -\frac{\gamma}{r^3}X \tag{44}$$

$$q^2Zp + Wq - (X + V)p \times q = A \tag{45}$$

Finally, the motor acceleration f is given by Eq. (15) in the form

$$X\left(3 - \frac{5X^2}{r^2}\right)f = -2(q^2Y + U)(2q^2Y + U) - 6q^2XV - \left(1 - \frac{5X^2}{r^2}\right)v^2 + \frac{\gamma}{r} - \frac{11\gamma}{r^3}X^2 + \frac{10}{r^2}XS\left(U + q^2Y - \frac{SX}{r^2}\right) \tag{46}$$

where

$$S = v \cdot r = XU + q^2(YV + ZW) \tag{47}$$

and r, v are given by Eqs. (25) and (26), respectively.

Equation (45) can be solved for q in the following manner: Taking the vector product of both sides with p, we find that

$$Wp \times q + (X + V)q = p \times A \tag{48}$$

Then, substituting for $p \times q$ from Eq. (45), we reach the result

$$\dot{p} = q = \frac{(X + V)p \times A + W(A - q^2Zp)}{W^2 + (X + V)^2} \tag{49}$$

This is a first-order differential equation for the primer vector p.

Integration Procedure

The integration process can now proceed thus. We first establish admissible starting values for all of our variables in the following manner: Choose initial values for the rocket coordinates x_i $(i = 1,2,3)$ relative to a fixed Cartesian frame F coinciding with R at the initial instant $(t = 0)$. Denoting these values by (a_1, a_2, a_3), we have initially

$$X = a_1, \qquad qY = a_2, \qquad qZ = a_3 \tag{50}$$

The initial value of r is now derived from Eq. (25) and then the initial value of q from Eq. (42) (note that the condition $3X^2/r^2 < 1$ must be satisfied). The initial values of Y and Z now follow from Eqs. (50) immediately. The initial value of W is next chosen arbitrarily and the corresponding initial values of U, V are then calculated from Eqs. (43) and (44), respectively. The initial components of p, q, $p \times q$ in the frame F are (1,0,0), (0,q,0), (0,0,q). Thus, by Eq. (45), the components of the vector A in F are given by

$$A_1 = q^2Z, \qquad A_2 = Wq, \qquad A_3 = -q(X + V) \tag{51}$$

using initial values for q, X, etc., already found. As the motion develops, these components of A in the fixed frame F will remain constant. This completes the calculation of initial values.

Denoting the components of p, q in F by p_i, q_i $(i = 1,2,3)$, respectively, and taking scalar products of the two members of Eq. (45) with p, q, $p \times q$, we arrive at the equations

$$q^2Z = A_1p_1 + A_2p_2 + A_3p_3 \tag{52}$$

$$q^2W = A_1q_1 + A_2q_2 + A_3q_3 \tag{53}$$

$$-q^2(X + V) = \begin{vmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} \tag{54}$$

Also, Eq. (49) can be resolved into three components giving the derivatives $\dot{p}_1, \dot{p}_2, \dot{p}_3$. These equations, together with Eqs. (35-37) and (41), can then be integrated for X, Y, Z, W, p_1, p_2, p_3 using a Runge-Kutta algorithm, the auxiliary variables r, q, U, V being regarded (throughout the process) as functions of X, Y, Z, W, p_1, p_2, p_3 determined by Eqs. (25), (43), (44), and (52). Equation (46) fixes f, and the q_i are calculable from Eq. (49). By including the equation

$$\dot{C} = f \tag{55}$$

in the Runge-Kutta integration process, the characteristic velocity C [Eq. (2)] can be derived at all stages of the maneuver, the initial value of C at its commencement being taken as zero.

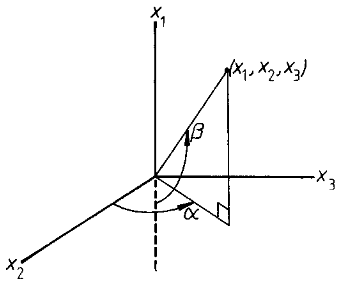

Fig. 1 Definition of the angles α and β.

The rocket coordinates in F follow from Eq. (23). Thus,

$$x_1 = Xp_1 + Yq_1 + Z(p_2q_3 - p_3q_2) \tag{56}$$

etc. Similarly, the components of rocket velocity are calculable from Eq. (24) by this formula

$$v_1 = Up_1 + Vq_1 + W(p_2q_3 - p_3q_2) \tag{57}$$

as well as other formulas.

A running check on the integration process can be made by verifying the satisfaction of Eqs. (7), (42), (53), and (54). In addition, if desired, Eqs. (39) and (40) can be included in the Runge-Kutta procedure to provide values for U and V that may be checked against those already found from Eqs. (43) and (44). If, after any integration step, these checks are found to be unsatisfactory, the program may be terminated or, alternatively, the size of the integration step for the variable t may be reduced until concordance is re-established.

Since the motor thrust must always be aligned with $\boldsymbol{p}$, f is an essentially positive quantity. If after an integration step has been completed it is discovered that f has become negative, the program must be terminated.

Family of Trajectories

Since the initial values of the four quantities x_1, x_2, x_3, W have to be chosen before the integration process can proceed, the algorithm described in the preceding section will generate a quadruple infinity of trajectories. However, as remarked earlier, any point on each of these trajectories can be treated as the initial point at which the maneuver commences, and by choosing this point according to some specific rule, we can reduce the assemblage of curves to a triply infinite set. It is natural, and we have found it convenient, to locate the commencement of a trajectory at a point where $f = 0$; if, then, as t increases, f becomes negative, the initial values must be adjusted as explained later. A further reduction to a doubly infinite set can, as explained earlier, be effected by fixing a scale factor. This can be managed by choosing the initial value of r as our unit of length; the coordinates x_i then have to satisfy the initial condition $x_1^2 + x_2^2 + x_3^2 = 1$ and the number of arbitrary parameters has been reduced by one. At the same time, we shall choose our unit of time so that the gravitational constant γ takes the unit value.

It is now helpful to introduce angles α, β, such that the initial values of the coordinates x_i can be expressed thus

$$x_1 = -\cos\beta, \quad x_2 = \cos\alpha \sin\beta, \quad x_3 = \sin\alpha \sin\beta \tag{58}$$

(see Fig. 1; we may assume that $0 < \alpha < 360$ deg, $0 < \beta < 180$ deg.) It is easily verified that if our equations are satisfied by functions $X(t)$, $Y(t)$, etc., then they are also satisfied by the same functions with the signs of $Z(t)$, $W(t)$ reversed. Thus, we need only compute trajectories for which the initial value of Z is positive (i.e., $x_3 > 0$) and can restrict α to the interval (0, 180 deg). Further, referring to Eq. (42), we see that since $r = 1$, $X = x_1$, initially, it is necessary that the initial value of x_1 satisfy the inequality $3x_1^2 < 1$. Thus, $\cos^2\beta < 1/3$ and 54.7 deg $< \beta <$ 125.3 deg.

Table 1 Orbital elements and characteristic velocities

	ℓ	e	ω	λ	i	C
$\alpha = 11$						
	0.73684	0.67185	−84.926	82.098	22.627	
	0.43395	0.77239	−110.110	90.771	25.860	0.14956
	0.44445	0.77472	−108.556	90.538	25.395	0.14925
$\alpha = 12$						
	0.70726	0.68921	−83.718	79.394	22.227	
	0.45382	0.76786	−105.607	86.881	24.771	0.12711
	0.46296	0.76994	−104.251	86.647	24.419	0.12692
$\alpha = 13$						
	0.67773	0.70683	−82.199	76.456	21.816	
	0.45593	0.77241	−101.826	83.233	23.833	0.11305
	0.46405	0.77428	−100.594	82.985	23.592	0.11293
$\alpha = 14$						
	0.64823	0.72468	−80.332	73.239	21.398	
	0.44800	0.78174	−98.347	79.553	23.097	0.10375
	0.45668	0.78311	−97.004	79.208	22.815	0.10366
$\alpha = 15$						
	0.61871	0.74277	−78.067	69.686	20.980	
	0.43280	0.79428	−94.962	75.690	22.377	0.09795
	0.44382	0.79481	−93.247	75.137	22.068	0.09787
$\alpha = 16$						
	0.58914	0.76114	−75.340	65.723	20.569	
	0.41141	0.80935	−91.525	71.515	21.705	0.09523
	0.42387	0.80932	−89.537	70.785	21.394	0.09516
$\alpha = 17$						
	0.55947	0.77981	−72.066	61.258	20.178	
	0.38405	0.82666	−87.889	66.874	21.072	0.09561
	0.40093	0.82538	−85.198	65.761	20.732	0.09554
$\alpha = 18$						
	0.52961	0.79889	−68.134	56.176	19.829	
	0.35020	0.84629	−83.873	61.544	20.485	0.09958
	0.36766	0.84475	−80.928	60.230	20.174	0.09950
$\alpha = 19$						
	0.49948	0.81850	−63.401	50.324	19.555	
	0.30853	0.86858	−79.173	55.132	19.968	0.10820
	0.32712	0.86664	−75.814	53.524	19.694	0.10809
$\alpha = 20$						
	0.46893	0.83886	−57.683	43.514	19.411	
	0.25679	0.89413	−73.151	46.838	19.609	0.12350
	0.27650	0.89175	−69.248	44.847	19.405	0.12334
$\alpha = 21$						
	0.43775	0.86029	−50.749	35.510	19.490	
	0.19187	0.92372	−64.205	34.730	19.757	0.14937
	0.21252	0.92079	−59.634	32.318	19.709	0.14908
$\alpha = 22$						
	0.40564	0.88335	−42.324	26.028	19.960	
	0.11191	0.95733	−47.966	13.667	22.376	0.19383
	0.13116	0.95414	−43.380	11.544	22.655	0.19315
$\alpha = 23$						
	0.37215	0.90906	−32.093	14.755	21.142	
	0.03294	0.98807	−19.479	−25.151	48.580	0.27800
	0.04664	0.98531	−17.642	−23.185	46.455	0.27563

Numerical computation reveals that if W is positive, for each value of α in its interval (0, 180 deg), the value of f increases monotonically as β increases from its lower limit of 54.7 deg toward 90 deg. For $\beta = 54.7$ deg, $f < 0$ and for $\beta = 90$ deg, $f = +\infty$. Accordingly, there is a unique value of β within this range for which $f = 0$ and it is easy to write a computer program that will identify this value of β. Similarly, we can find a unique value of β, lying in the range (90, 125.3 deg), for which $f = 0$. Since the initial conditions when W is positive can always be amended to suit the case when W is negative (see the following discussion), we need not study this case further.

Thus, for each pair of values of W and α, we can calculate two values of β (one acute and one obtuse) and corresponding values of the x_i, to provide initial conditions from which the numerical integration of our equations can proceed with $f = 0$ at the initial point. The doubly infinite aggregate of trajectories generated in this way is labeled by the pair of parameters (W, α).

However, there remains one further restriction to be applied. We require that f should remain positive as the integration proceeds. We observe that if $X=X(t)$, $Y=Y(t)$, $Z=Z(t)$, $U=U(t)$, $V=V(t)$, $W=W(t)$, $f=f(t)$ are functions satisfying Eqs. (35–37) and (39–41), then these equations are also satisfied by $X=-X(t)$, $Y=-Y(t)$, $Z=Z(t)$, $U=-U(t)$, $V=-V(t)$, $W=W(t)$, $f=-f(t)$. Hence, if it is discovered that, when the integration procedure is initiated, f commences to take negative values, then the initial values of X, Y, U, V must be reversed in sign and the integration recommenced; this will generate a trajectory along which f is positive. This correction is equivalent to replacing α,β by their supplements. It follows that, provided this correction is always made when necessary, there is no loss of generality in restricting β to be acute, so that only one value of β is associated with each parameter pair (W,α).

It remains to treat the case $W<0$. First note that if $X=X(t),\ldots,f=f(t)$ satisfy Eqs. (35–37) and (39–41), with $f>0$ for $t>0$ and $f=0$ at $t=0$, then an alternative solution is $X=-X(-t)$, $Y=Y(-t)$, $Z=Z(-t)$, $U=U(-t)$, $V=-V(-t)$, $W=-W(-t)$, $f=-f(-t)$. Since $f(t)<0$ for $t<0$ [if we assume that $\dot{f}(t)\neq 0$ at $t=0$], this alternative solution will also give $f>0$ for $t>0$ and $f=0$ at $t=0$. To generate this alternative solution numerically, the appropriate initial conditions are $X=-X(0)$, $Y=Y(0)$, $Z=Z(0)$, $U=U(0)$, $V=-V(0)$, $W=-W(0)$. Thus, having computed a trajectory starting with a positive value of W and along which f is positive, we need only reverse the signs of the initial values of X, V, W (leaving the initial values of Y, Z, U unchanged) and repeat the integration procedure to generate a new trajectory along which f is positive. The initial value of W for this trajectory will be negative. In this way, we avoid the calculation of initial values in all cases where $W<0$.

After performing these integrations on a computer for a range of values of W and α, we have found that the behavior of f along a trajectory follows one of two possible patterns. Either f ultimately tends to $+\infty$ or f ultimately becomes zero (prior to assuming unacceptable negative values). In both cases, the trajectory terminates and is of finite length. f may increase monotonically as it approaches $+\infty$ or it may at first increase and later decrease to zero again. However, more complex patterns of behavior are possible, e.g., with $W=0.5$, $\alpha=10$ deg, it is found that f at first increases to a maximum, then decreases to a minimum, finally increasing monotonically to $+\infty$. In the case where the trajectory lies in a plane (corresponding to zero initial values of Z and W), it is been shown[3] that the trajectory is a spiral of infinite length and f increases monotonically from 0 to $+\infty$ as the spiral winds into the center of attraction from infinity.

Comparison with Optimal Two-Impulse Transfer

Taking $W=0.5$ and permitting α to range over values from 11 to 23 deg with a 1-deg interval, we have found that in all such cases the behavior of f is the same, i.e., it first increases monotonically from zero to a maximum and subsequently decreases monotonically back to zero. We have computed the orbital elements at the two ends of each of these trajectories and then compared the characteristic velocity for the transfer between these terminal orbits with that corresponding to the optimal two-impulse transfer between the same pair of orbits. The results (to five significant figures) are presented in Table 1. The orbital elements have been chosen as follows: The x_1x_2 plane of the frame F has been treated as the reference plane; this is intersected by the orbital plane in the line of nodes. Then, ℓ = semilatus rectum; e = eccentricity; ω = longitude of the periapse, measured in the sense of description of the orbit from the ascending node; λ = angle between the x_1 axis of F and the line from the origin to the ascending node; and i = inclination of the orbital plane to the reference plane $0x_1x_2$ (this angle is taken in the first quadrant if the orbit's projection onto the reference plane is described in the positive sense, and in the second quadrant otherwise). C is the characteristic velocity. For each value of α, the first two rows show the elements of the terminal orbits and the last row the elements of the most economical two-impulse transfer orbit. All angles are expressed in degrees.

It will be noted that transfer by two impulses is in all cases more economical than transfer via the IT arc, but that the benefit is comparatively small. No comparison has been made between the times of transfer by the two modes.

Conclusions

For the problem of optimizing the trajectory of a rocket in an inverse square law gravitational field with respect to the propellant expenditure, the singular extremal continuous thrust arcs form a two-parameter family. The numerical computation of members of this family is most conveniently performed in a rotating reference frame aligned with the motor thrust. Although the arcs are nonoptimal, their characteristic velocities are generally within 1% of the best two-impulse transfer.

References

[1]Lawden, D. F., *Optimal Trajectories for Space Navigation,* Butterworths, London, 1963, pp. 54–64.

[2]Lawden, D. F., "Rocket Trajectory Optimzation: 1950–1963," *Journal of Guidance, Control, and Dynamics,* Vol. 14, No. 4, 1991, pp. 705–711.

[3]Lawden, D. F., "Optimal Intermediate-Thrust Arcs in a Gravitational Field," *Astronautica Acta,* Vol. 8, No. 2, 1962, pp. 106–123.

[4]Kopp, R. E., and Moyer, H. G., "Necessary Conditions for Singular Extremals," *AIAA Journal,* Vol. 3, No. 8, 1965, pp. 1439–1444.

[5]Robbins, H. M., "Optimality of Intermediate Thrust Arcs of Rocket Trajectories," *AIAA Journal,* Vol. 3, No. 6, 1965, pp. 1094–1098.

[6]Kelley, H. J., Kopp, R. E., and Moyer, H. G., "Singular Extremals," *Topics in Optimization,* edited by G. Leitmann, Academic, New York, 1967, Chap. 3.

[7]Gurley, J. G., "Optimal-Thrust Trajectories in an Arbitrary Gravitational Field," *SIAM Journal on Control,* Vol. 2, No. 3, 1964, pp. 423–432.

[8]Goh, B. S., "The Second Variation for the Singular Bolza Problem," *SIAM Journal on Control,* Vol. 4, No. 2, 1966, pp. 309–325.

[9]Robbins, H., "Optimal Rocket Trajectories with Subarcs of Intermediate Thrust," 27th Congress of the International Astronautical Federation (Madrid), Oct. 1966.